

P. Grinevich, A. Mironov, S. Novikov ¹
 On the Nonrelativistic 2D Purely Magnetic
 Supersymmetric Pauli Operator

The Introduction. In general, the motion of charged particle in the external electromagnetic field is described by the full 4D Dirac equation (for the spin equal to $1/2$). For time-independent fields Pauli obtained a famous beautiful nonrelativistic approximation L^P (see [1], paragraph 33). We are going to consider a special 2D case where the electric field is equal to zero. The remarkable theory of this operator starts in 1979–80. In the present work we found unification of the Theory of Ground States for the Pauli Operator with the Algebro-Geometric Spectral Theory of periodic scalar Schrödinger operators and corresponding theory of soliton nonlinear systems. Especially analog of the 2D Burgers hierarchy plays a key role here. The Aharonov–Bohm type terms in magnetic field like the quantized delta function also play important role in our construction. This work extends results published in [2, 3, 4]. We pay special attention to the self-adjoint boundary problems allowing to reveal physical meaning of the Bloch functions with nonunitary multipliers. Such ideas were known for $D = 1$ since 1974, but for $D > 1$ there extension never was discussed.

History: The Ground level of magnetic Pauli operators. In 1979–1980 three groups of authors studied the ground level using ”the Factorization Property” of the 2D purely Magnetic Pauli Operator written in the Lorenz gauge $A_{1x} + A_{2y} = 0$ with $A_1 = i\Phi_y, A_2 = -i\Phi_x$: (Avron–Seiler [AS], [5], Aharonov–Casher [AC], [6], Dubrovin and Novikov [DN] [7, 8]:

¹P. Grinevich, Landau Institute, Moscow, e-mail pgg@landau.ac.ru, A. Mironov, Sobolev Math Institute, Novosibirsk, and Laboratory of Geometric Methods in Mathematical Physics, Moscow State University e-mail mironov@math.nsc.ru, S. Novikov, University of Maryland, College Park and Landau/Steklov Institutes of RAS, Moscow, e-mail novikov@ipst.umd.

edu,

Homepage www.mi.ras.ru/snovikov (click publications), the items 177, 178, 179

The Pauli Operator is a second order 2-component Schrödinger-type operator. We use a special system of units such that

$L^P = L^+ \oplus L^-$ and $-L^\pm = (\partial_x + i\Phi_y)^2 + (\partial_y - i\Phi_x)^2 \pm \Delta\Phi$, acting on the space of vector-function $\Psi = (\Psi^+, \Psi^-)$.

We assume that the charge is also equal to 1. Following formal observation is extremely useful:

Lemma 1 *Let $Q = \partial_z - A_z$, $Q^+ = -(\partial_{\bar{z}} + A_{\bar{z}})$. The Scalar Operators L^\pm are Strongly Factorized $L^+ = QQ^+$, $L^- = Q^+Q$, $\partial_z = \partial = \partial_x - i\partial_y$, $A_z = A_1 - iA_2$, magnetic field $B = \Delta\Phi$.*

The proof can be easily obtained by the direct substitution.

The most interesting classes of magnetic fields are [AC] and [DN]. Following results were obtained (see proof below):

1. AC: Rapidly decreasing fields,
 $||B|| = |\int \int_{\mathbb{R}^2} B dx dy| < \infty$. The ground states form a finite-dimensional space of dimension $m \in \mathbb{Z}$,
 $m \leq [B]/2\pi < m + 1$.

2. DN: Arbitrary periodic fields with integer-valued flux through the elementary cell $\frac{1}{2\pi} \int \int_{cell} B dx dy = m \in \mathbb{Z}$.

The ground states generate an infinite dimensional subspace in the Hilbert Space $L_2(\mathbb{R}^2)$ isomorphic to the Landau level. The so-called "Magnetic-Bloch states" form a manifold isomorphic to the total space of the holomorphic vector bundle over the 2-torus; The projection exactly coincides with the so-called "magnetic quasimomentum". The first Chern class of this bundle is equal to the basic cohomology class in the group $H^2(T^2, \mathbb{Z})$, the fibre is isomorphic to \mathbb{C}^m .

Remark 1 *The experience of Differential Topology after Whitney, Pontryagin and Thom led S. Novikov to the idea to study "Generic" operators in order to define their topological invariants. It started after this work and allowed to reveal topological invariants of operators in the space of quasimomenta. In particular Chern numbers of the transversal dispersion relations were invented and studied in 1980–81 by S. Novikov and A. Lyskova-see [9, 10, 11, 12]. This set of ideas was partly rediscovered by physicists of the Thouless group few years later to explain the famous "Integral Quantum Hall Phenomenon" experimentally discovered at that time.*

Theorem 1 *In the cases AC and DN all ground states are the Instantons belonging to one spin-sector only: It means precisely following*

a. They satisfy to the 1st order equations $Q^+\psi = 0$ in the sector L^+ for the case $[B] > 0$ and $Q\psi = 0$ in the sector L^- for the case $[B] < 0$. It is a simple prototype of the self-duality equation.

b. They belong to the Hilbert Space $L_2(\mathbb{R}^2)$.

Proof. a. The case AC. Let Ψ is a zero mode for the operator $L^+\Psi = 0$ and $\Psi \in L_2$. The standard "Instanton argument" is $0 = \langle QQ^+\Psi, \Psi \rangle = \langle Q^+\Psi, Q^+\Psi \rangle$ for the ground states. So $Q^+\Psi = 0$. Every square-integrable solution of this equation $\Psi \in L_2$ defines a zero mode of the operator L^+ and vice versa. So our result follows if we will be able to find solutions from the Hilbert Space $L_2(\mathbb{R}^2)$. Let $0 < [B] < \infty$, $m \leq \frac{1}{2\pi}[B] < m+1$ in the case AC (above). We look for Ψ as a product of two factors: one is an arbitrary holomorphic polynomial $P_l, l = 0, \dots, m$, other one depends on magnetic field:

$$\Psi_l = P_l(z) \exp\{-R(x, y)\}$$

with special solution to the equation $\Delta R = B$ if $[B] > 0$:

$$R(x, y) = -\frac{1}{4\pi i} \int \int_{\mathbb{R}^2} \ln |z - w| dw \wedge d\bar{w}.$$

The growth of R obviously depends on the value of magnetic flux $[B]$:

$$\int \int_{\mathbb{R}^2} |\Psi_l|^2 dx \wedge dy < \infty$$

for $l = 0, \dots, m-1$. So we found all ground states in the case AC. There are no square integrable solutions in the second sector L^- .

b. The case DN. Solutions here also can be found in one spin-sector only. As above, we assume that the magnetic flux through elementary cell is a positive integer $m \in Z_+$. We consider rectangular lattice with periods $2\omega \in \mathbb{R}, 2i\omega' \in i\mathbb{R}$. Let us define R by the formula

$$R = -\frac{1}{4\pi i} \int \int_K \ln |\sigma(z - w)| B(w, \bar{w}) dw \wedge d\bar{w}$$

where $\Delta R = B$, K is elementary cell. We look for the solutions written in the form

$$Q^+\Psi = 0, \Psi = \exp\{-R(x, y)\} [e^{az} \prod_{j=1, \dots, m} \sigma(z - a_j)] \lambda$$

where $\lambda \neq 0$ is any number.

Lemma 2 *These solutions are the Magnetic Bloch functions, i.e. the eigenvectors of magnetic translations $T_j^*, j = 1, 2$:*

$$T_1^* \Psi(x, y) = \Psi(x + 2\omega, y) e^{-if_1(x, y)} = \varkappa_1 \Psi(x, y),$$

$$T_2^* \Psi(x, y) = \Psi(x, y + 2\omega') e^{-if_2(x, y)} = \varkappa_2 \Psi(x, y).$$

Vector potential is chosen in the Lorentz gauge form $A = (i\Phi_y, -i\Phi_x)$ for $B = \Delta\Phi$ and $\Phi = -R$ as above. By definition,

$$A(x + 2\omega, y) = A(x, y) + i(f_{1x}, f_{1y}), A(x, y + 2\omega') = A(x, y) + i(f_{2x}, f_{2y}).$$

Proof. We use the standard transformation properties

$$\sigma(w + 2\omega) = -e^{-2\eta(w+\omega)} \sigma(w), \sigma(w + 2i\omega') = -e^{-2i\eta'(w+i\omega')} \sigma(w)$$

where $\zeta(w) = \sigma_w/\sigma, \eta = \zeta(\omega), i\eta' = \zeta(i\omega')$ for the rectangular lattice. Our theorem immediately follows from this properties by the direct calculation. After calculations, we are coming to the following formula for such values of parameters that both multipliers are unitary $|\varkappa_1| = |\varkappa_2| = 1$:

$$\text{Re}a = \text{Re}\{\eta_1/\omega[2 \sum_j a_j - 1/\pi \int \int_K zB(x, y)dx dy]\},$$

$$\text{Im}a = \text{Im}\{\eta'/\omega'[2 \sum_j a_j - 1/\pi \int \int_K zB(x, y)dx dy]\}.$$

By definition, the components of quasimomenta are p_1, p_2 :

$$e^{2ip_1\omega} = \varkappa_1, e^{2ip_2\omega'} = \varkappa_2.$$

They have a form following from the formulas for the magnetic Bloch functions above:

$$p_1 + m\pi/2\omega = \text{Im}(a - \eta/\omega \sum_j a_j),$$

$$p_2 + m\pi/2\omega' = \text{Re}(a - \eta'/\omega' \sum_j a_j).$$

The quasimomentum map is well-defined on the space of parameters describing the states with unitary multipliers. It looks like

$$p_1 + ip_2 = p(a_1, \dots, a_m) = -(2\pi i/|K|) \sum_j a_j + \text{const}$$

where $|K| = 4\omega\omega'$ is an area of the elementary cell K . Its image is a torus T^2 , the fibre is a complex space \mathbb{C}^m .

The elementary arguments presented in the work [7, 8] show that this family describes all eigenstates of the operator L^P nearby of the ground level $\epsilon = 0$. It follows from the properties of the elliptic operator L^P with fixed unitary multipliers. Its Kernel is exactly m -dimensional because the index of operator Q^+ with $Q^+ : \Psi^+ \rightarrow \Psi^-$ is equal to m , and the adjoint operator $Q : \Psi^- \rightarrow \Psi^+$ has no Kernel. No singularities appear in this family for all points in the torus of quasimomenta (i.e. for all unitary multipliers). So we conclude that there is a finite nonzero gap Δ separating zero level from all other levels for the operator L^P .

Supersymmetry and Laplace Transformations.

The operator $S = Q^+ : \Psi^+ \rightarrow \Psi^-$ and $S : \Psi^- \rightarrow 0$ is called **Super-Symmetry** for L^P . Here $S^2 = 0, SL^P = L^P S, S^* L^P = L^P S^*$. The "adjoint" supersymmetry operator is $S^* = Q : \Psi^- \rightarrow \Psi^+$, $S^*(\Psi^+) = 0$. We have $SS^* + S^*S = L^P$.

It implies that all higher levels are 2-degenerate (the ground level is ∞ -degenerate). The Super-Symmetry operator was invented here in 1980s.

It is in fact a partial case of the so-called Laplace Transformations known since XVIII Century for the scalar 2D Schrödinger operators. Every 2nd order operator of the (Hyperbolic) form $L = \partial_x \partial_y + A \partial_x + B \partial_y + V$ can be presented in the form $L = (\partial_x + B)(\partial_y + A) + W = QQ' + W$. We define

$$L \rightarrow \tilde{L} = WQ'W^{-1}Q + W, \Psi \rightarrow \tilde{\Psi} = Q'\Psi.$$

We conclude that $L\Psi = 0$ implies $\tilde{L}\tilde{\Psi} = 0$ and

$$\tilde{L} = (\partial_y + A - W_y/W)(\partial_x + B) + W = (\partial_x + B)(\partial_y + A - W_y/W) + \tilde{W}$$

where $\tilde{W} = W - A_x + B_y + (\ln W)_{xy}$.

We are going to deal with Elliptic Case important for Quantum Mechanics, and replace ∂_x by $\partial = \partial_z$ and ∂_y by $\bar{\partial}$. So we have $L = QQ^+ + W \rightarrow \tilde{L} = WQ^+W^{-1}Q + W$ and $\Psi \rightarrow Q^+\Psi = \tilde{\Psi}$ with $Q = \partial + A, Q^+ = -\bar{\partial} + \bar{A}$. **For the special case of factorizable operators $W = \text{const}$ this transformation acts on the whole spectrum of operator L . Physicists**

identify it with "supersymmetry" for the Pauli Operator: $L\Psi = \epsilon\Psi$ implies $\tilde{L}\tilde{\Psi} = \epsilon\tilde{\Psi}$ for $\tilde{\Psi} = Q^+\Psi$.

The gauge invariants of operator $L = QQ^+ + W$ are Magnetic field $2B = A_{\bar{z}} - \bar{A}_z$ and Potential W . It is convenient to write Laplace transformations in terms of invariants only:

$$\tilde{W} = W + \tilde{B}, \quad \tilde{B} = B + 1/2\Delta \log W.$$

It was observed that the whole infinite chain of Laplace Transformations is equivalent to the "2D Toda Lattice System". This observation is going back to the XIX Century school of Darboux (see [13]). This school also started to consider the Cyclic Chains of Laplace Transformations and pointed out that they lead to several fundamental completely integrable systems. These works contains only formal calculations dedicated to the hyperbolic case (see historic discussion in [14]).

In the work of S. Novikov with A. Veselov (1997) cyclic chains were considered in the elliptic case where global restrictions play fundamental role. For smooth elliptic double-periodic operators it was proved in [14] that corresponding Schrödinger operators L are always Algebro-Geometric, i.e. the collection of nonsingular complex Bloch-Floquet solutions to the equation $L\Psi = 0$ form an algebraic curve (i.e. of finite genus). However, this theory contains only Topologically Trivial operators where all coefficients are double periodic, and magnetic flux through the elementary cell is equal to zero. In order to include topologically nontrivial operators, another type of Laplace chains was considered in [14]. Let $L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_n$ be a Laplace chain such that the end operators L_0, L_n are both strongly factorizable up to constant:

$$L_n = Q_n Q_n^+, \quad L_0 = Q Q^+ + \text{const.}$$

The ground level is highly degenerate here for the smooth double periodic magnetic field B_n . For $n = 2$ nontrivial cases were found such that some higher level is also infinitely degenerate. Probably, no more than 2 levels may be highly degenerate except the case of Landau operator in the homogeneous magnetic field.

Algebro-Geometric (AG) Scalar Schrödinger operators.

Question: Is Theory of Ground States for the Purely Magnetic Pauli

Operator related somehow to the Algebro–Geometric Theory of the scalar periodic 2D Schrödinger Operators based on the Selected Energy Level and 2D Soliton Hierarchies?

Let us remind here the history of this subject.

History: The Algebro–Geometric Theory of the 2D Second Order Scalar Schrödinger Operators and Corresponding Soliton Hierarchies based on the selected energy level was started in 1976 by Manakov [15] and Dubrovin, Krichever, Novikov [16].

The magnetic field is always Topologically Trivial in this theory (i.e. coefficients of vector-potentials are double periodic). The idea to use following systems instead of ordinary Lax pairs was invented in [15]

$$\frac{dL}{dt} = (LH - HL) + fL.$$

Nontrivial example with both operators L, H of order 2 was demonstrated showing that this new approach is nonempty. Long time this specific system was not studied. A "2-point scalar Baker–Akhiezer function" was invented in [16] for the solution of inverse problems for L based on one energy level. It allows to construct the whole hierarchy of higher systems associated with the original one. In particular, the whole hierarchy is completely different from the KP-type situation: it depends on 2 infinite sets of times t'_j, t''_j associated with 2 "infinite" points. Our Riemann surface here is a family of all nonsingular complex Bloch–Floquet functions of one selected energy level $L\psi = 0$ ("**the Complex Fermi Curve**"). In the Algebro–Geometric Case it has finite genus. Two "infinite" points are selected on this algebraic Fermi curve.

Some important time-invariant Reductions (i.e. invariant subsystems or sub-hierarchies) were actively studied in 1980s. Several authors found them either for Nonlinear Systems or for Inverse Spectral (Scattering) Data (or for both). Solution of this problem for the Inverse Data is more difficult: It implies in particular the description of all reduced hierarchy. However, the existence of time-invariant reduction is much easier to observe on the level of equation.

Our Main Goal here is Quantum Mechanics and Spectral Theory. We use Nonlinear Systems as a tool for this goal.

1. The Data leading to the Self-Adjoint Periodic Operators were found by Cherednik in 1980 [17].
2. The Data leading to the traditional operators $L = -\Delta + U(x, y)$ (with

zero magnetic field) were found by Veselov and Novikov in 1984 (see [18, 19, 20, 21]).

Extension of these results to the Rapidly Decreasing Potentials was studied in the works of S. Manakov, P. Grinevich, R. Novikov and S. Novikov in the late 1980s (see [22, 23, 24]). Krichever in [25] proved that every 2D smooth periodic potential can be approximated by the AG ones.

The Problem solved recently in our work: **Calculate AG Data for the Reduction leading to the Factorized Operators**

Solution of the Reduction Problem.

Consider a very first Manakov's System $L_t = [H, L] + fL$ where both L, H are second order operators: $L = \partial_x \partial_y + G \partial_y + S, H = \Delta + F \partial_y + A$,

$$G_t = G_{xx} - G_{yy} + (F^2/4)_x - (G^2)_x - A_x + 2S_y,$$

$$S_t = S_{yy} - S_{xx} - 2(GS)_x + (FS)_y,$$

with differential constraints

$$F_x = 2G_y, \quad A_y = 2S_x, \quad f = 2G_x - F_y.$$

Konopelchenko already pointed out in 1988 (see [26]) that the reduction $S = 0$ is time-invariant for this system. It looks like 2D analog of the famous Burgers system.

We believe in the following **Informal Principle**: Every natural time-invariant reduction of completely integrable Soliton system can be effectively described in terms of the scattering (spectral) data and Riemann surfaces.

How to describe this specific reduction for the Inverse Spectral Data? Is it possible? Making replacement $x, y \rightarrow z, \bar{z}$ we are coming to elliptic operators most interesting for us.

Our recent result [2, 3, 4] describes corresponding Inverse Problem Data for the operator L with $S = 0$: Take Riemann Surface (the Complex Fermi Curve) splitted into nonsingular pieces $\Gamma = \Gamma' \cup \Gamma''$ with genres g', g'' . They cross each other $P_j = Q_j, P_j \in \Gamma'', Q_j \in \Gamma', j = 0, \dots, l$. (see Fig 1 for the self-adjoint elliptic case where $g' = g''$).

Take infinities $\infty_1 \in \Gamma', \infty_2 \in \Gamma''$ with local parameters k'^{-1}, k''^{-1} . Construct function $\psi = (\psi', \psi'')$ with asymptotic $\psi' \sim c(x, y)e^{k'\bar{z}}(1 + O(k'^{-1}))$, $\psi'' = e^{k''z}(1 + O(k''^{-1}))$ and divisors of poles D', D'' of degree $g' + l, g''$ not crossing infinities and intersection points. In the hyperbolic case take the variables x, y instead of z, \bar{z} . Time dynamics can be added in the standard way:

$$\psi' = e^{k'\bar{z} + \sum_{s>1} (k')^s t'_s} (c + O(k'^{-1})), \quad \psi'' = e^{k''z + \sum_{s>1} (k'')^s t''_s} (1 + O(k''^{-1})).$$

The crossing property has a form

$$\psi'(Q_j) = \psi''(P_j), j = 0, 1, \dots, l.$$

Theorem 2 *Such Data generate a 2-point Baker–Akhiezer function $\psi = (\psi', \psi'')$ on the splitted surface $\Gamma = \Gamma' \cup \Gamma''$ and scalar operator $L' = \Delta + G\partial_{\bar{z}}$ with $S = 0$ such that $L'\psi' = L'\psi'' = 0$.*

The Data for Self-Adjoint Operators. For the selection of self-adjoint case we need to have $g' = g''$ and to add the degenerate Cherednik-type restriction. One can say that it is simply a limit of Cherednik condition for the degenerate Riemann Surface $\Gamma = \Gamma' \cup \Gamma''$. We take infinite points $\infty' \in \Gamma'$ with local parameter k'^{-1} and $\infty'' \in \Gamma''$ with local parameter k''^{-1} . An anti-holomorphic map should be given $\tau : \Gamma'' \rightarrow \Gamma'$ such that $\tau(\infty'') = \infty'$, $\tau^*(k') = \bar{k}''$, and τ permutes the intersection points $\tau(P_j) = Q_{k_j}$. The inverse map $\Gamma' \rightarrow \Gamma''$ we also call τ , so we have $\tau : \Gamma \rightarrow \Gamma, \tau^2 = 1$. The divisor of poles is $D' \in \Gamma'$ and $D'' \in \Gamma''$ with degrees $|D'| = g' + l, |D''| = g''$ where $g' = g''$ and poles do not cross the intersection points and infinities. The total divisor of poles at the unified Riemann Surface Γ is $D = D' \cup D''$. We require as usually that there exists a meromorphic one-form Ω with simple zeroes in the points of divisors $D, \tau(D)$ and simple poles in the infinite points ∞_1, ∞_2 . The divisor (K) consists of the zeroes of any holomorphic one-form, \sim is a "linear equivalence" of divisors:

$$D + \tau(D) \sim (K) + \infty' + \infty''.$$

Now let our surface be degenerate. The one-form Ω also became degenerate. Its degeneration consists of two meromorphic one-forms ω', ω'' in the Riemann surfaces Γ', Γ'' with simple poles in the crossing points $P_j \rightarrow \tau(Q_j)$ such that $\text{res}_{P_j} \omega'' + \text{res}_{Q_j} \omega' = 0$. Their divisors should satisfy to the conditions:

$$(\omega') = D' + \tau(D'') - \infty' + \sum_j Q_j, \quad (\omega'') = D'' + \tau(D') - \infty'' + \sum_j P_j.$$

After the non-unitary gauge transformation

$$L' \rightarrow L = \frac{1}{\sqrt{c}} L' \sqrt{c}, \quad \psi \rightarrow \frac{\psi}{\sqrt{c}}$$

in Γ', Γ'' , we obtain a self-adjoint operator

$$L = QQ^+ = (\partial - A)(-\bar{\partial} - \bar{A}), \quad A = \frac{1}{2} \bar{\partial} \log c.$$

Taking $L^+ = L$ and $L^- = Q^+Q$, we construct a Purely Magnetic Pauli Operator

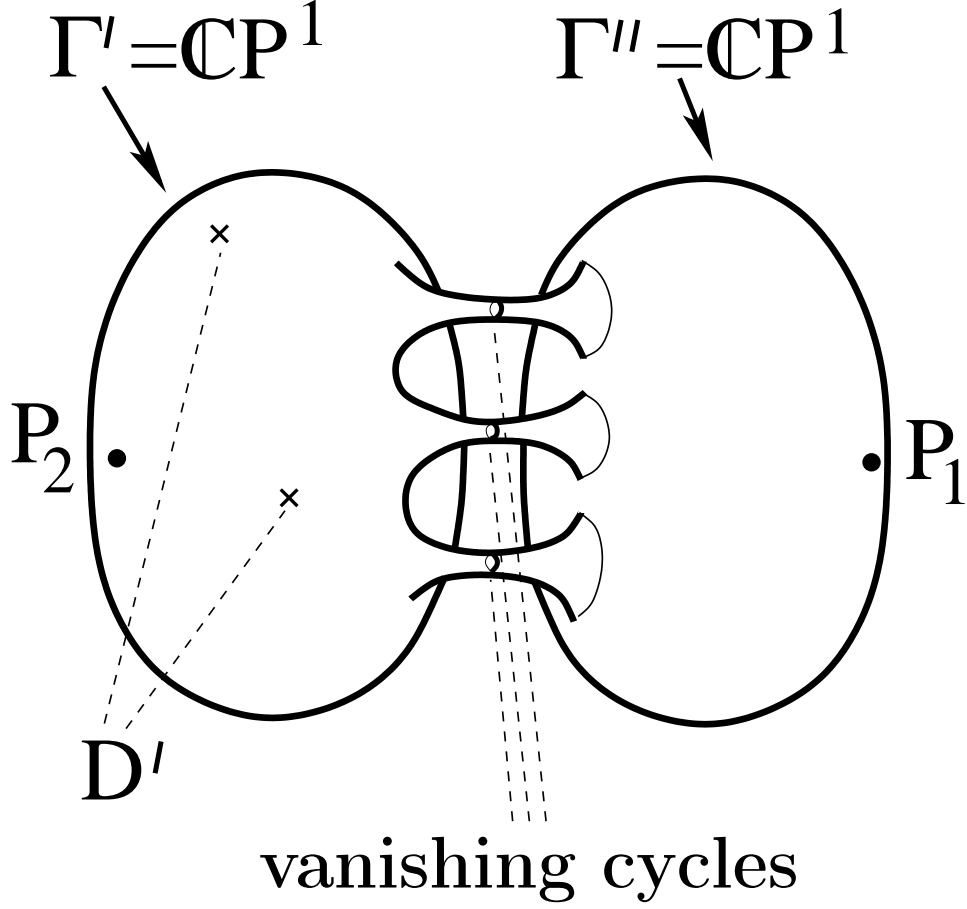
$L^P = QQ^+ \oplus Q^+Q$. The Magnetic Field is real $B = 1/2\Delta \log c$, periodic or quasiperiodic and **topologically trivial**. It is nonsingular if $c \neq 0$, so the operator is self-adjoint in this case.

How to find ground states in the Hilbert space? It is very simple in the periodic nonsingular case $c \neq 0$. Take $\psi_0 = c^{1/2}$ in the first spin-sector L^+ because $Q^+\psi_0 = 0$.

Take $\phi_0 = c^{-1/2}$ in the second sector L^- because $Q\phi_0 = 0$. **In the case of periodic smooth real $c \neq 0$ we have exactly two periodic ground state functions located in both sectors. They present the bottom of the CONTINUOUS SPECTRUM near the ground level $\epsilon = 0$.** See below full description of all complex nonsingular Bloch functions of the ground level. Their relationship to the magnetic Bloch functions found in [7] in 1980 for the topologically nontrivial magnetic field also will be discussed below.

The Case of Genus zero (Fig 1)

Fig 1



We take $l + 1$ intersection points presented as $k' = k_s$ and $k'' = p_s$ in Γ', Γ'' , and divisor $D' = (a_1, \dots, a_l)$ of degree l in Γ' . We have

$$\Psi' = e^{k'\bar{z}} \frac{w_0 k'^l + \dots + w_l}{(k' - a_1) \dots (k' - a_l)}, \Psi'|_{k'=k_s} = e^{p_s z}.$$

As we can see, $c = w_0$. So $c = \sum_{s=0}^l \kappa_s e^{W_s(z, \bar{z})}$, where $W_s = p_s z - k_s \bar{z}$ (see [3]).

Lemma 3 *For every trigonometric/exponential polynomial c of the form above with any set of linear forms W_s and complex coefficients κ_s there exists AG Data generating such function c .*

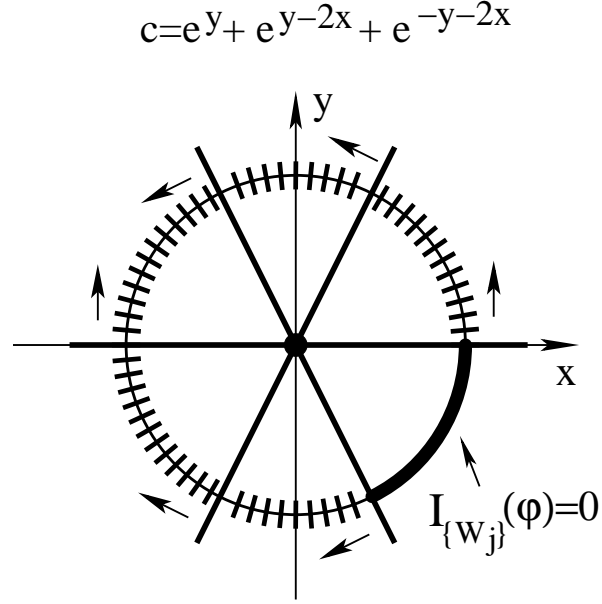
Let $W_s = \alpha_s x + \beta_s y, (\alpha_s, \beta_s) \in C_W^2$.
Transformation $c \rightarrow c' = ce^{\gamma + \alpha x + \beta y}$ leads to the gauge equivalent operator (with the same magnetic field).

There exist 3 types of Real Solutions:

1. Purely Exponential Positive Case ("The Lump-type fields")
 $\kappa_s > 0, (\alpha_s, \beta_s) \in \mathbb{R}^2$.
2. Periodic Trigonometric Real Case. It will be considered below jointly with the case $g = 1$.
3. Mixed exponential/trigonometric case. It can be realized only if its "dominating part" belongs to the case 1. So we will not discuss it.

The case 1. Let "the Tropical Sum" of the forms in the set $\{W\}$ is nonnegative $I'_{\{W\}}(\phi) = \max_s(\alpha_s \cos \phi + \beta_s \sin \phi) \geq 0$.
Then $c^{-1/2}$ is bounded in \mathbb{R}^2 .
For the angles $I'_{\{W\}}(\phi) > 0$ we have a rapid decay
 $c^{-1/2} \rightarrow 0, R \rightarrow \infty$, Let $I(\phi) = \max\{I'(\phi), 0\}$.

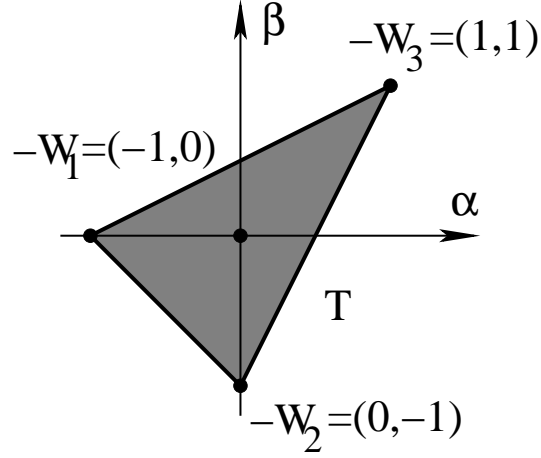
Fig 2a



In every class $c' \in ce^W, W' \in R_W^2$, the set of representatives c' with nonnegative $I = I'_{\{W'\}}(\phi) \geq 0$ forms a convex polytop \bar{T}_c . Its inner part $T_c \subset \bar{T}_c$ consists of all c' such that $I_{\{W'\}} > 0$. Open part T_c is always nonempty for $l > 2$. \bar{T}_c is nonempty for $l > 1$. (see Fig 2b for $l = 3$)

Fig 2b

b) $\mathbb{R}^2 = (\alpha, \beta)$
 $W_j = \alpha_j x + \beta_j y$



Here $e^y + e^x + e^{-y-x} = c$

Magnetic field is decaying for $R \rightarrow \infty$ except some selected angles, it is a Lump Type Field analogous to the KP "Lump Potentials". A linear sum under the $1/2\Delta \log()$ reflects the Linearization of the Burgers Hierarchy in the variable c .

$$[B] = \int \int_{D_R^2} B dx dy = -1/2R \oint_{S^1} I_{\{W\}}(\phi) d\phi + O(R^{-1})$$

**All points in T_c define ground states in the Hilbert Space $L_2(\mathbb{R}^2)$.
The boundary points define the bottom of continuous spectrum.**

The Periodic Problem.

Let lattice in \mathbb{R}^2 be rectangular and $z = x + iy$. For every real periodic function c we can define a whole family of "possible" meromorphic Bloch functions

$$\psi''_{ext,\pm} = f(z)(\sqrt{c})^\pm e^{uz - \zeta(p)z} \frac{\sigma(z + p + R)}{\sigma(z + R)}$$

where $f(z)$ is an arbitrary elliptic function.

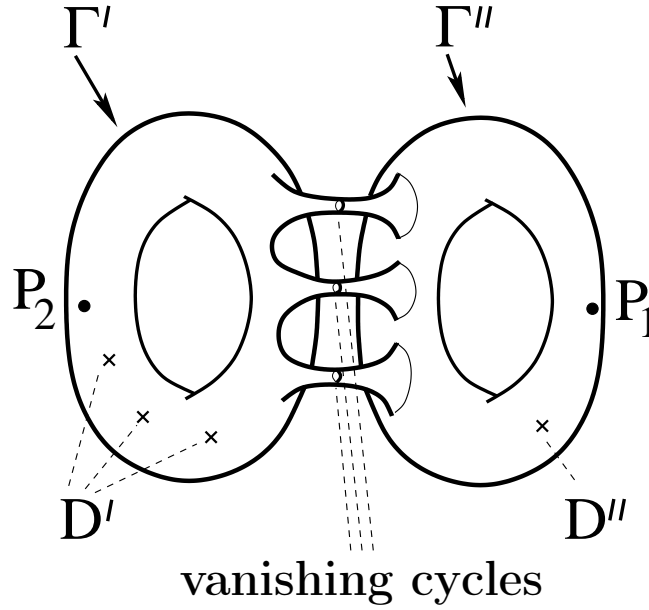
We have $Q^+ \psi''_{ext,-} = 0$ for $L = L^+ = QQ^+$. For anti-holomorphic functions we get $Q \psi''_{ext,+} = 0$ for $L^- = Q^+Q$.

Let $c \neq 0$. We need only nonsingular functions, so our manifold is $u \in \mathbb{CP}^1 = \Gamma''$ with function $\psi''_+ = e^{uz} \sqrt{c}$ (or $e^{u\bar{z}} \sqrt{c}$).

Let c has an isotropic zero. We have larger family of nonsingular (or "weakly singular") ψ''_+ -Bloch functions because $\sqrt{c}/\sigma(z + R)$ became only weakly singular such that it may enter spectrum. So the full manifold is $M^2 = \mathbb{CP}^1 \times \Gamma$ where Γ is an elliptic curve. If c is a trigonometric polynomial, $1/c$ is not. We have c for L^+ , so we have $c' = 1/c$ for L^- . So we need to calculate ψ' for all real periodic smooth functions c' .

The case of genus 1.

Fig 3



We take elliptic curve $\Gamma' = \Gamma'' = \mathbb{C}/\Lambda$ with Euclidean local parameters k, p (the point 0 is "infinity"), periods $2\omega \in \mathbb{R}, 2i\omega' \in i\mathbb{R}$, n intersection points $Q_0, Q_1, \dots, Q_n \in \Gamma'$ and $R_0, \dots, R_n \in \Gamma''$. Divisors $D' = (P_1, \dots, P_n), D'' = P$ have degree $n+1, 1$ correspondingly. We have

$$\psi' = e^{-\bar{z}\zeta(k)} \frac{\prod_s \sigma(k - Q_s)}{\prod_l \sigma(k + P_l)} \times \left(\sum_j w_j \frac{\sigma(k + \bar{z} + \tilde{P} + \tilde{Q} - Q_j)}{\sigma(k - Q_j)} \right).$$

Here $\tilde{P} = P_1 + \dots + P_n, \tilde{Q} = Q_0 + \dots + Q_n$, sum as in \mathbb{C}

$$\psi'' = e^{-z\zeta(p)} \frac{\sigma(p + z + P)}{(\sigma(z + P)\sigma(p + P))}, \quad \psi'(Q_s) = \psi''(R_s).$$

All singularity of the quantity c disappear after multiplication

$$\tilde{c} = c\sigma(\bar{z} + \tilde{Q} + \tilde{P})\sigma(z + P).$$

Take $n = 1, Q_0 = -Q_1, R_0 = Q_1, R_1 = Q_0$ and solution to the equation $\omega\zeta(Q_0) = \eta_1 Q_0$. We have $P = \tilde{Q} + \tilde{P}$ in this case, so $-1/2\Delta|\sigma|^2 = -2\pi\delta(z)$.

So **the Conclusion** based on the case $g = 1$ is following:

The least singular real Data lead finally to magnetic field $\tilde{B} = -1/2\Delta\tilde{c}$ which is periodic nonsingular with magnetic flux equal to ONE QUANTUM UNIT. The original AG topologically trivial magnetic field $B = 1/2\Delta c$ extracted from our Inverse Spectral Data is always singular for $g = 1$; it has total magnetic flux equal to zero through the elementary cell and quantized δ -singularity at the phone of nonsingular field \tilde{B} , located in the point P . So this field corresponds to the "Aharonov-Bohm" (AB) situation.

For $g > 1$ the number of quantized δ -functions is equal to $k > 1$. **We can get nonsingular AG operators only from genus 0.** Both pieces of the original Riemann surface $\Gamma = \Gamma'' \cup \Gamma'$ are presented in the form of k -sheeted branching covering over elliptic curve $\Gamma'' \rightarrow \Gamma_0$ for $g > 1$ as it was in the works of Krichever dedicated to the elliptic KP (see [27]).

The important Conclusion is following: Comparison with [7] shows that the Quantized δ -flux does not affect spectrum nearby of the zero level.

The complex Bloch-Floquet manifolds (consisting of nonsingular or weakly singular Bloch functions admitted by the spectral problem) for the level $\epsilon = 0$ and genus $g = 1$ is

$$M = M^2$$

with function

$$\psi_{ext,-}'' = \frac{1}{\sqrt{c}} \left(e^{uz} \times e^{-\zeta(p)z} \frac{\sigma(z+p+R)}{\sigma(z+R)} \right),$$

$$L^+ \psi_{ext}'' = L^+ \psi' = 0.$$

Let us point out that both functions ψ', ψ'' are originally singular here $g = 1$; Our next step is to reduce operator to the self-adjoint form L . For that we have to multiply both ψ', ψ'' by the factor $1/\sqrt{c}$. Let us start with ψ'' . The function $\psi_{ext}'' 1/\sqrt{c}$ is "nonsingular", i.e. it is singular weakly enough for entering spectrum in $L_2(\mathbb{R}^2)$ for the operator L with the quantized Aharonov–Bohm term. After singular unitary gauge transformations $L \rightarrow \tilde{L}$ all singularity of this function disappear. So it enters the smooth spectrum of the operator \tilde{L} with magnetic field \tilde{B} where singularity is removed. Now about ψ' . The function ψ'/c is nonsingular but the function ψ'/\sqrt{c} is singular. After the singular unitary gauge transformation it maps into function $\tilde{\psi}'/\sqrt{c}$ with simple pole. It satisfies to the equation

$$\tilde{L}(\sqrt{\sigma/\bar{\sigma}}(\psi'/\sqrt{c})) = 0.$$

So we conclude that the family $\tilde{\psi}'/\sqrt{c}$ of magnetic Bloch functions for the operator \tilde{L} with nonsingular topologically nontrivial magnetic field has a pole in the variables (x, y) .

The function ψ'/c is nonsingular. It satisfies to the equation $L_0(\psi'/c) = 0$ in all nonsingular points where $L_0 = \partial(\bar{\partial} + \Phi_z)$. At the same time we have $\Phi_z \sim 1/\bar{z}$ near singularity, all other terms are smooth. So the function $f = (\bar{\partial} + \Phi_z)(\psi'/c)$, $c = e^\Phi$, are antiholomorphic almost everywhere. However, in the point P where singularity is located, f has a form $f \sim m(k)/\bar{z}$. Therefore $\psi'/c \sim m(k) + \bar{z}O(1)$ near this point. We conclude that $L_0(\psi'/c) = m(k)\delta_P(x, y)$ because $\partial(1/\bar{z}) = (const)\delta$.

Reconsider now the case $g = 0$ comparing it with $g = 1$.

For $c \neq 0$ and $g = 0$ the Bloch manifold is equal to the union $\Gamma'' \cup \Gamma'$, and both are \mathbb{CP}^1 ; Let c has an isolated zero (minimum) which is isotropic. Magnetic field became singular, with δ -term. The extended instanton Bloch function $\psi_{ext,+}''$ became only weakly singular and capable to serve spectrum for the operator L^+ . It depends on the points of the complex 2-manifold M^2 , and all possible values of the complex multipliers

$$M^2 \rightarrow (\kappa_1, \kappa_2) \in \mathbb{C}^* \times \mathbb{C}^*$$

are presented in this family. The energy dispersion relation is degenerate here $\epsilon : M^2 \rightarrow 0$.

We have $M^2 = \mathbb{CP}^1 \times \Gamma_0$ where Γ_0 is an elliptic curve,

$$\psi''_{ext,+} = (const(u))e^{p\bar{z}-\zeta(u)\bar{z}}\sigma(\bar{z}+u)\sqrt{c}/\sigma(\bar{z}).$$

Its singularity is in fact like $\sqrt{\bar{\sigma}/\sigma}$ because c has a lattice of double zeroes like $|\sigma|^2$. Make singular gauge transformation

$$\psi''_{ext,+} \rightarrow \sqrt{\sigma/\bar{\sigma}}\psi''_{ext,+}$$

(here σ depends on \bar{z} under the square root). This function transforms into the nonsingular magnetic Bloch function of the operator L^P with topologically nontrivial magnetic field \tilde{B} . Magnetic Bloch multipliers after transformation are equal to the original standard Bloch multipliers.

Our Conclusion is that the periodic case $g = 1$ gives result similar to the special case of $g = 0$ where c has an isolated isotropic zero, interchanging sectors \pm (i.e. z and \bar{z}). **The higher number $k \geq 1$ of isotropic zeroes for $g = 0$ leads to the "higher rank" family of weakly singular instanton type Bloch functions M^{k+1} .** Removing δ -singularities by the singular gauge transformations we get smooth magnetic Bloch manifold M^{k+1} corresponding to the periodic magnetic field like in [7] with higher flux.

We know that the algebro-geometric case simply corresponds to the case of trigonometric polynomials. We take rectangular lattice in the plane x, y . Following relation is true

$$Q^+\psi' = M(k)\sqrt{c}e^{\bar{z}k}.$$

Now we choose normalization of ψ' such that $M(k) = k$. Let us remind that Q^+ is "the Super-symmetry Operator" in periodic case.

Let us extend our results to the "infinite" trigonometric series for $g = 0$.

We use for that the formula

$$\psi' = k \sum_j \left[\frac{\kappa_j e^{p_j z - k_j \bar{z}}}{k - k_j} \right] e^{k\bar{z}}.$$

We are dealing here with the new normalization: here k_j are simply the lattice points.

As a corollary we have

$$\sum_j \kappa_j e^{p_j z - k_j \bar{z}} = c$$

Apply this result to the infinite series $c \rightarrow c' = 1/c$. It gives us a function ψ'_- for the second component L^- of the Pauli operator.

The Boundary Problems.

Problem: The component Γ' of the Bloch–Floquet manifold does not affect the ordinary spectrum in the Hilbert space of functions in the whole plane \mathbb{R}^2 . Can we use it for solving physically meaningful (i.e. self-adjoint) boundary problems?

Let us remind following: For one-dimensional self-adjoint periodic Schrödinger operators $L = -\partial_x^2 + u, u(x+T) = u(x)$, the 2-sheeted Riemann surface Γ of the Bloch–Floquet function $L\psi = \lambda\psi, T^*\psi = \psi(x+T) = \varkappa\psi$, is such that all branching points $a_s, s \geq 0$, are real. They are the boundaries separating the spectral zones from the forbidden zones (gaps) at the real line. In order to reconstruct potential, we need to know also the position of poles of $\psi(\lambda, x, x_0)$ for $\lambda \in \Gamma$ (normalized by condition $\psi(x_0) = 1$). They are the points on the Riemann surface $\gamma_j \in \Gamma$ forming together a "Divisor D ". There is exactly one pole inside of every finite gap for the smooth potentials:

$$\gamma_j(x_0) \in [a_{2j-1}, a_{2j}].$$

The zeroes of ψ are located exactly in the points $\gamma_j(x)$. Therefore our Bloch–Floquet function $f(x^*) = \psi(\gamma_j(x), x^*)$ satisfies to the Dirichlet boundary problem at the interval $x^* \in [x, x+T]$ where T is a period because $f(x) = f(x+T) = 0$.

Another interpretation follows from the fact that the same function $f(x^*)$ decreases to the right or to the left from the zero $f|_{x^*=x} = 0$ corresponding to the sheet of Riemann surface where the pole $\gamma(x_0)$ is actually located. So the function $f(x^*)$ belongs to the discrete spectrum on the corresponding half-line $x^* \in [x, +\infty]$ or $x^* \in [-\infty, x]$.

Question: Can we find similar realization for the Bloch functions ψ', ψ'' for the periodic smooth topologically trivial magnetic field $B = 1/2\Delta \ln c$ where c is real and nonzero?

The classical John von Neumann theory of self-adjoint boundary conditions (as I.Gelfand pointed out to S.Novikov many years ago, about 1971) follows to the scheme quite similar to the Novikov's idea to construct the "Hermitian K-theory" over rings with involution: Self-adjoint extensions of symmetric operators correspond to Lagrangian subspaces in the Hamiltonian module over the ring of functions. Let $\gamma \subset T^2$ be a contour in the torus presenting the boundary of our domain (or its connected component).

Consider first the "scalar" boundary conditions for the operator $L = L^+ = QQ^+$ or $L = L^- = Q^+Q$ separately, not mixing different components. We reduce the integral to the boundary

$$\int \int_D [(L\psi)\bar{\phi} - \psi(\bar{L}\bar{\phi})]d^2x = \oint_{\gamma} [\psi_1\bar{\phi}_2 - \psi_2\bar{\phi}_1]dt$$

where $\psi_1(t) = \nabla_n \psi|_{\gamma} = (\partial_n + A_n)\psi|_{\gamma}$ and $\psi_2 = \psi|_{\gamma}$. Here n means the vector normal to the boundary component (external). To make self-adjoint extension means to select a "Lagrangian Subspace" $(\psi_1(t), \psi_2(t)) \in \Lambda$ for the skew hermitian boundary form such that

$$\oint_{\gamma} [\psi_1\bar{\psi}'_2 - \psi_2\bar{\psi}'_1]dt = 0$$

for every pair of elements in the subspace Λ . There are following types of boundary conditions.

The "Ultralocal" boundary conditions such that no derivatives along the directions tangent to the boundary are involved. They are elliptic.

1. The Dirichlet b.c $\psi_2 = \psi|_{\gamma} = 0$.
2. The Neuman b.c. $\psi_1 = \nabla_n \psi|_{\gamma} = 0$.
3. The Leontovich b.c. $\alpha(t)\psi_1 + \beta(t)\psi_2 = 0$ with (α, β) real and nonzero for all $t \in \gamma$. Only ratio $(\alpha(t)/\beta(t))^{\pm 1}$ is invariant here. This b.c. has a topological charge (the degree of map $\gamma = S^1 \rightarrow S^1 = \mathbb{R}P^1$).

The General "Local" b.c.. They are nonelliptic.

Let a pair of scalar differential operators on the circle U, V be given such that one of them is invertible (for example, equal to 1). Our b.c. is $U\psi_1 = V\psi_2$. This b.c. is self-adjoint if $(UV^+)^+ = UV^+ = VU^+$. In the special case then one of them is equal to 1, another one is an arbitrary self-adjoint differential operator on the boundary.

An interesting special case we have if $U = 1$, and V is a first order operator $V = i\alpha\partial_t + v(t)$, $\alpha, v(t) \in \mathbb{R}$. The b.c. of the form $\psi_1 = \nabla_n \psi = \pm i\psi_t + v(t)\psi$ we call the **"d-bar b.c." or d-b.c.** The reason for that is following: we can write this b.c. (for the sign $-$) in the form

$$[Q^+ \psi - e^{i\theta(t)} v(t) \psi]_\gamma = 0$$

where $Q^+ = -(\partial_{\bar{z}} - A_{\bar{z}})$ and $A_1 = i\Phi_y, A_2 = -i\Phi_x, A_{\bar{z}} = -\Phi_{\bar{z}}, \theta(t)$ is an angle rotating the coordinate frame x, y to the Frenet frame n, ∂_t . The d-bar b.c. is self-adjoint but not ultralocal. It is nonelliptic. We could not find any traces of it in the literature. However, it is very closed to the elliptic b.c. and self-adjoint. One can imagine that such b.c. might appear in the problem of physics. Replacing the operator V by $V' = +i\partial_t + v(t)$ we obtain similar **"d- b.c."** where Q^+ is replaced by $Q = \partial - \Phi_z$. It has similar properties.

There are also the General Nonlocal Self-Adjoint b.c.. We do not consider here nonlocal boundary conditions.

Boundary Conditions for Pauli operators mixing components.

The operator L^P acts on vector-functions $\Psi = (\psi^+, \psi^-)$. After calculations we obtain

$$\langle L^P \Psi, \Phi \rangle - \langle \Psi, L^P \Phi \rangle = \oint_\gamma \{[\psi_1^+ \bar{\phi}_2^+ - \psi_2^+ \bar{\phi}_1^+] + [\psi_1^- \bar{\phi}_2^- - \psi_2^- \bar{\phi}_1^-]\} dt$$

where

$$\psi_1^\pm = \nabla_n \psi^\pm|_\gamma, \psi_2^\pm = \psi^\pm|_\gamma.$$

The Ultralocal b.c.. They are elliptic.

Our skew Hermitian form has signature $2, 2$ in every point of boundary $t \in \gamma$. The family of all Lagrangian subspaces is isomorphic to the group U_2 for every $t \in \gamma$. So all collection of ultralocal self-adjoint b.c. can be classified by the smooth maps $S^1 \rightarrow U_2$ with topological charge from the group $\pi_1(U_2) = \mathbb{Z}$. Lagrangian subspaces in every point $t \in \gamma$ may have 3 types:

1. Subspaces of the form

$$\psi_1^\alpha = \sum_\beta R^{\alpha, \beta} \psi_2^\beta$$

where matrix R is Hermitian and $\alpha, \beta = \pm$.

2. Similar subspaces with numbers 1 and 2 interchanged $\psi_2 = R\psi_1$.

The intersection of the types 1 and 2 is exactly a subclass such that Hermitian matrix R is nondegenerate.

The types 1 and 2 can be unified by the form $U\psi_1 = V\psi_2$ where one of matrices U, V is invertible and $UV^+ = VU^+$ is Hermitian.

3. Subspaces of the form $a\psi_1^+ + b\psi_1^- = 0, c\psi_2^+ + d\psi_2^- = 0$. Lagrangian property implies $a\bar{c} + b\bar{d} = 0$. We can normalize these equations reducing them to the form $|a|^2 + |b|^2 = 1, |c|^2 + |d|^2 = 1$. We are coming to the arbitrary matrix from the group SU_2 numerating such special Lagrangian subspaces (or to the arbitrary unit vector $(a, b) \in \mathbb{C}^2$ determining the whole matrix).

The General Local b.c. mixing components.

We can write a natural differential analog of the scalar types above in the form

$$U\psi_1 = V\psi_2, \psi_1 = \nabla_n \psi, \psi_2 = \psi$$

where one of two operators U, V on the circle is invertible. Our requirement is $UV^+ = VU^+$.

The most interesting for us are the following b.c where no second derivatives of Ψ in the boundary points are involved.

$$\begin{aligned} \nabla_n \psi^+ &= A^+ \psi^+ + b \nabla_n \psi^- \\ -\psi^- &= c \psi^+ + d \nabla \psi^- \end{aligned}$$

where $A^+ = \alpha^+ i \partial_\tau + a$ are the first order operators along the boundary. Here the functions $\alpha^+(t), a(t), d(t) \in R, b(t), c(t)$ are smooth and $c = \bar{b}$.

This b.c are nonelliptic if order of operator A^+ is more than zero. The ultralocal case corresponds to the zero order $\alpha^+ = 0$. We are going to discuss the following

Problem. Let $\Psi = (\psi^+, \psi^-)$ be a formal zero mode solution to the Pauli operator $L^P \Psi = 0$. For which contours it satisfies to some self-adjoint boundary condition?

We are going to consider below this Problem both in mixing and nonmixing cases.

Our intention is to show that for some special mixing and non-mixing b.c. these contours can be described as a trajectories of dynamical system (foliation) in the 2-torus T^2 or in the covering space R^2 . Consider first the Ultralocal case.

Theorem 3 *Let $d = 0$ and $\alpha^+ = 0$. The coefficients a, c, b are such that $a \in \mathbb{R}$ and $c = \bar{b}$ if and only if the boundary curve $\gamma = \partial D$ is a leaf of foliation $\omega = 0$ where*

$$\omega = (|\psi^+|^2 + |\psi^-|^2)d\Phi + |\psi^+|^2 * d\theta^+ + |\psi^-|^2 * d\theta^-$$

*Here $\psi^\pm = |\psi^\pm| \exp\{i\theta^\pm\}$ and $*d\Phi$ is a vector potential of magnetic field, star $*$ is a standard duality operator on the complex plane.*

The case $b = 0$ leads to the nonmixing b.c.. We will discuss this case below in details. For the d -bar problem we have $\alpha^+ = -1$. Assuming as above that $d = 0$ we are coming to the similar dynamical system. It is investigated below in details for the nonmixing case $b = c = 0$.

The proof of this theorem is not complicated. It can be obtained by the elementary manipulation.

The case which is most interesting for us is $\psi^+ = \psi'$ and $\psi^- = \lambda\psi''$ where λ is a parameter, and $\psi'' = Q^+\psi'$. Here Q^+ is the operator which is treated in quantum theory as a supersymmetry for L^P (see above). Our dynamical system depends on λ . For $\lambda = 0$ we have a Nonmixing Case.

This pair defines a General Bloch Solution of the zero energy level to the Physical Pauli Operator with some (nonunitary) multiplier like in the one-dimensional case where we had Bloch solutions in the forbidden bands as eigenfunctions of the self-adjoint boundary problem..

Let us make some very general remarks concerning the **Nonmixed Boundary Problems**:

Let a contour $\gamma \subset T^2$ be given with following Domain $D = T^2 \setminus \gamma$. Find solution to the boundary problem 1 or 2 such that $L^+\psi = 0$ in D and

The Ultralocal Problems. Let $\alpha(t)\nabla_n\psi + \beta(t)\psi = 0$ at the contour γ . Consider first the Problem 1 with **contours homotopic to zero in T^2** (see Fig 4 a), $\partial D = \gamma$. Solution of the "Leontovich type" boundary problem ψ satisfying to this relation should be found from the integral equation

$$\psi = \int \int_{\mathbb{R}^2} [p(k)\psi' + q(k)\psi''] d^2k.$$

For the Problem 2 with **contours γ nonhomotopic to zero in T^2** we have $\partial D = \gamma \cup \gamma'$ (see Fig 4 c). There is a natural \mathbb{Z} -covering $\hat{D} \rightarrow D$ which is a strip in \mathbb{R}^2 with average direction $g = [\gamma] \in H_1(T^2, \mathbb{Z}) = \mathbb{Z}^2$ (see Fig 4

b). The element $g \in \mathbb{Z}^2 \subset \mathbb{C} = \mathbb{R}^2$ acts freely $g : \hat{D} \rightarrow \hat{D}$ as a shift by the complex number $g : z \rightarrow z + g, \bar{z} \rightarrow \bar{z} + \bar{g}$. By definition, the components of boundary

$$\partial\hat{D} = \hat{\gamma} \bigcup \hat{\gamma}'$$

are coverings over γ, γ' . g can be viewed as a vector of lattice \mathbb{Z}^2 in \mathbb{R}^2 . Another generator $g' \in \mathbb{Z}^2$ complementary to g maps exactly $g' : \gamma \rightarrow \gamma'$. We are saying that $\gamma'(\hat{\gamma}')$ **is located to the right from** $\gamma(\hat{\gamma})$ if $|\varkappa(g')| < 1$. Let $c \neq 0$. All functions $\psi'(k), \psi''(k)$ with k orthogonal to g have unitary Bloch multipliers $|\varkappa(g)| = 1$ along the shift g . We have 4 domains D_+, D_-, \hat{D} in \mathbb{R}^2 and D in T^2 (see Fig 4):

$$D_+ \bigcup D_- = \mathbb{R}^2, \quad D_+ \bigcap D_- = \hat{\gamma}, \quad g'(D_-) \subset D_-, \quad g'(D_+) \bigcap D_- = \hat{D}.$$

We require $\alpha(t)\nabla_n\psi + \beta(t)\psi = 0$ on the contours γ, γ' **but allow to have different pairs of coefficients $(\alpha, \beta), (\alpha'\beta')$ for γ and γ' for the domain \hat{D} such that both of them enter the boundary**. The function ψ should be constructed essentially from ψ' but we can add also ψ'' if necessary: The most general possibility is that ψ is given by the integral along the k -axis orthogonal to g :

$$(I) : \psi = \int_{k \geq 0} [\psi'(k, x, y)p(k) + \psi''(k, x, y)q(k)]dk$$

for the half-plane $D_+ \subset \mathbb{R}^2$

$$(II) : \psi = \int_{k \leq 0} [\psi'(k, x, y)p(k) + \psi''(k, x, y)q(k)]dk$$

for the half-plane $D_- \subset \mathbb{R}^2$

$$(III) : \psi = \int_{k \in \mathbb{R}} [\psi'(k, x, y)p(k) + \psi''(k, x, y)q(k)]dk$$

for the strip $\hat{D} \subset \mathbb{R}^2$

$$(IV)_\varkappa : \psi_\varkappa = \sum_{m \in \mathbb{Z}} [\psi'(k_m, x, y)p_m(\varkappa) + \psi''(k_m, x, y)q_m(\varkappa)]$$

for $D \subset T^2$, where corresponding k_m are such that all $\psi'(k_m), \psi''(k_m)$ have the same fixed unitary multiplier $\varkappa = e^{k_m \bar{g}}$ (i.e. $k_m = k_0 + 2\pi im/\bar{g}$ where $m \in \mathbb{Z}$).

The local "d-bar" (or "d-") type Boundary Problems.

As above in the Ultralocal case we have two types of contours :

Problem 1 with contours γ homotopic to zero in T^2 where $\gamma = \partial D$.

Problem 2 with contours $\gamma \subset T^2$ non-homotopic to zero in the torus; here $\partial D = \gamma \cup \gamma'$.

For the second case (most interesting for us) we consider the same type of domains D_+, D_-, \hat{D}, D with boundaries non-homotopic to zero in T^2 and their coverings in \mathbb{R}^2 as above (see Fig 4).

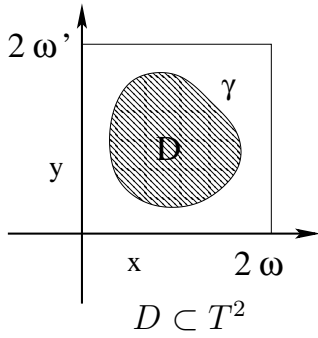


Fig 4a

γ is homotopic to 0

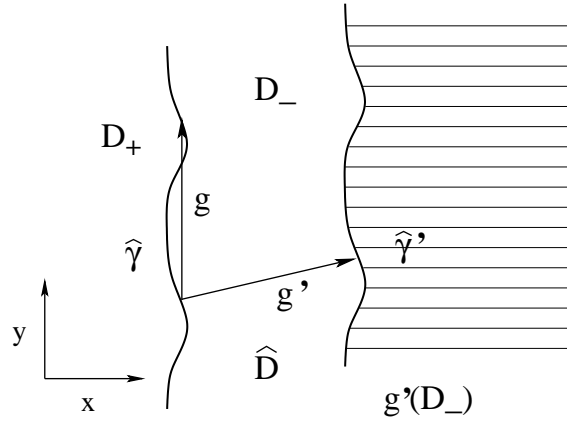
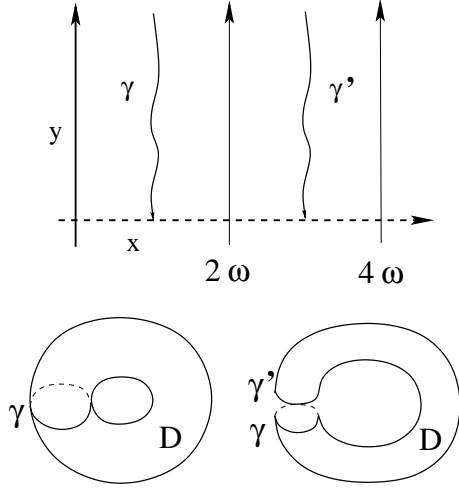


Fig 4b

γ is not homotopic to 0

Fig 4a and Fig 4b are associated with Example 2 below. Here g is parallel to the average direction of $\hat{\gamma}$



$$D = T^2 \setminus \gamma, \quad \partial D = \gamma \cup \gamma'$$

Fig 4c

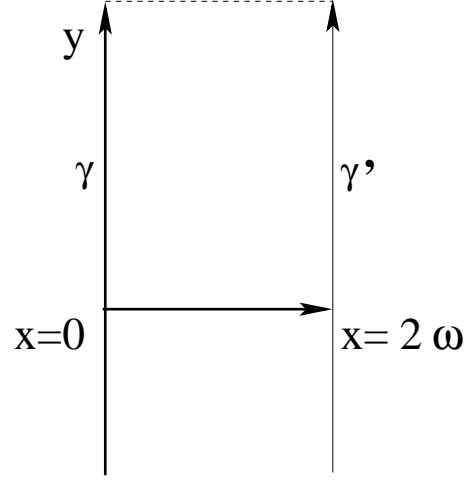


Fig 4d

Fig 4c and Fig 4d are associated with Example 1. The contour $\hat{\gamma}$ is simply $x = 0, -\infty < y < \infty$ in Fig 4c.

Our b.c. relation along the boundary is (with possibly different coefficients in the different components of boundary)

$$Q^\pm \psi = v(t) \psi e^{i\theta(t)}$$

or

$$\nabla_n \psi = (\pm i \partial_t + u(t)) \psi$$

for real u, v . We can try to find solution ψ using the integral equations as above.

We already discussed above the following question: **which general restrictions should be satisfied for the class of functions ψ solving the Leontovich type problem in the magnetic field with vector-potential $A = -i * d\Phi = i(-\Phi_y dx + \Phi_x dy)$?**

We consider now only nonmixing case. Assuming that $\psi = \rho(x, y) e^{i\theta(x, y)}$ we can see that the ratio $\nabla_n \psi / \psi$ should be real. Here n is a normal vector to the boundary contour $\gamma \subset \partial D$. We are coming to the following statement (a partial case of the theorem above):

Lemma 4 *The boundary contour γ for the self-adjoint scalar operator $L\psi = \lambda\psi$ with any Leontovich type b.c. should be a leaf of foliation given by the equation $*(d\theta + A/i) = \Omega = 0$ or $(\theta_y + \Phi_x)dx + (-\theta_x + \Phi_y)dy = 0$ if the phase function $\theta(x, y)$ is known. Vice versa, if the contour γ is known, the restriction $*d\theta = -d\Phi$ along the contour should be satisfied.*

The Special Contours where solution to the Nonmixing B.C. is a Bloch function ψ' .

Example 1: ψ' as a solution to the Leontovich Boundary Problem for the Special Contours of the first kind.

Let $g = 0$. We construct solution with $2n + 1$ crossing points p_j, k_j where

$$p_j = a + ib_j, \quad p_{n+j} = -p_j, \quad k_{n+j} = -k_j.$$

We have constants (nothing to do with Bloch multipliers) $\kappa_{n+j} = \kappa_j$ and all parameters a, b_j, κ_j are real. Let $\sum_{j=1}^n \kappa_j = 0$. We have

$$c = \kappa_0 + 2 \sum_{j=1}^n \kappa_j \cos(2(b_j x + ay)),$$

$$\frac{\psi'}{k} = e^{k\bar{z}} \left(\frac{\kappa_0}{k} + \sum_{j=1}^n \left(\kappa_j \frac{e^{p_j z - k_j \bar{z}}}{k - k_j} + \kappa_{n+j} \frac{e^{p_{n+j} z - k_{n+j} \bar{z}}}{k - k_{n+j}} \right) \right).$$

For $x = 0$ we have $c = \kappa_0 = \text{const}$. We call such contours **Special, of the first kind**. We take now contour $\gamma : x = 0$ or $c = \text{const}$ with boundary problem

$$\psi_n = \alpha(x)\psi.$$

Let us point out here that for such contours we have $\nabla_n = \partial_n$ because $d\Phi = 0$ at the contour $\Phi = \ln c = \text{const}$ and $A = (i\Phi_y, -i\Phi_x)$. We need to satisfy condition $\psi'_x/\psi \in \mathbb{R}$ for $x = 0$. Presenting ψ' in the form $\psi' = e^{k\bar{z}}(f + ig)$ we see that $\psi'_x/\psi|_{x=0} \in \mathbb{R}$ follows for the real $k \in \mathbb{R}$ from the requirements $g|_{x=0} = g_x|_{x=0} = 0$. After elementary calculations we see that $g|_{x=0} = A_1 \cos(ay) + A_2 \sin(ay)$ and $g_x|_{x=0} = B_1 \cos(ay) + B_2 \sin(ay)$ where the constants A_q, B_q are linearly expressed through the constants κ_j above. Our boundary condition is equivalent to 4 homogeneous linear equations

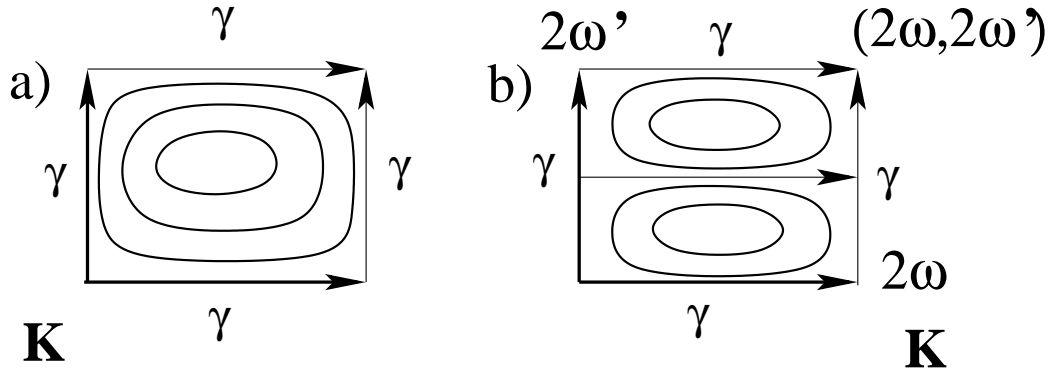
$A_1 = A_2 = B_1 = B_2 = 0$. So for $n > 5$ we can find solution for every fixed values of constants $a, b_j, k \in \mathbb{R}$. The real values of k are orthogonal to the direction of the contour $x = 0$. So we proved following

Lemma 5 *For every set of exponents with odd number $2n + 1$ of crossing points and $n > 5$, there exists a divisor (i.e. the set of constants κ_j) such that corresponding Bloch–Floquet function $\psi'(k, x, y)$ with real $k \in \mathbb{R}$ satisfies to the Leontovich b.c. $\psi'_x/\psi' \in \mathbb{R}$ for the contour $\gamma : x = 0$. The Bloch multiplier along the direction of contour is unimodular $\kappa = e^{-iky}$.*

So our problem is self-adjoint. Corresponding function $\psi'(k, x, y)$ serves as a solution to the Leontovich boundary problem in the domain \bar{D} : $[0 < x < 2\omega]$, and in one of two half-planes D_+ : $[0 < x]$ or D_- : $[x < 0]$ depending on sign of the corresponding value of k . It serves also for the spectrum in the domain D where $x \in [0, 2\omega]$ in the torus T^2 (i.e. periodic in the variable y), see Fig 4c and 4d.

Fig 5 below is associated with the next Example 2 dedicated to d-bar boundary conditions. It shows situation listed in the final conclusions as a "maximal" boundary contour homotopic to zero such that the area inside coincides with the whole area of the torus. In particular, the magnetic flux through this domain is equal to zero according to our assumptions on the class of magnetic fields under consideration. In this case a boundary cycle consists of separatrices (see Fig 5a and Fig 5b):

Fig 5



Example 2: ψ' as a solution to the "d-bar" boundary problem for the special contours of the second kind.

We write d-bar problem in the form $Q^+\psi = u(t)e^{i\theta(t)}\psi$ where $u \in \mathbb{R}$ and $\theta(t)$ is an angle from the frame x, y to the Frenet frame n, ∂_τ along the boundary curve γ . Let $\psi = \psi'(k, x, y)/\sqrt{c}$. We have $Q^+\psi/\psi = e^{i\theta_k(x,y)}u_k(x, y)$ where $u_k \in \mathbb{R}$. So our contour γ should be tangent to vector (director) field ∂_τ in the torus. This field depends on parameter $k \in \mathbb{C}$. It is obtained from the direction ∂_x by the rotation $e^{i\theta_k(x,y)}$ in the point x, y or $z = x + iy$. This vector (director) field can be described by the zeroes of one-form

$$\text{Re}[(Q^+\psi/\psi)d\bar{z}] = 0.$$

Let

$$c = 1 + \sum_j^n a_j e^{l_j \bar{z} - \bar{l}_j z} + \bar{a}_j e^{-l_j \bar{z} + \bar{l}_j z}$$

and in the new normalization where poles are located in the crossing points

$$\frac{\psi'}{k} = \left(\frac{1}{k} + \sum_j \left(a_j \frac{e^{l_j \bar{z} - \bar{l}_j z}}{k + l_j} + \bar{a}_j \frac{e^{-l_j \bar{z} + \bar{l}_j z}}{k - l_j} \right) \right) e^{k\bar{z}}.$$

Now we make a gauge transformation to the self-adjoint form $L = QQ^+$, and $\psi = \psi'/\sqrt{c}$. We have $Q^+\psi/\psi = -\bar{\partial}\psi'/\psi'$. But $\bar{\partial}(\psi'/k) = 2ce^{k\bar{z}}$ for the newly normalized ψ' . We obtain result $Q^+\psi/\psi = -2cke^{k\bar{z}}/\psi'$.

Therefore we described the set of contours for which d-bar boundary condition has $\psi'(k, x, y)$ as a solution. We call them **the Special Contours of the second kind**. They are leaves of the foliation on 2-torus given by the following equation:

$$(cke^{k\bar{z}}/\psi)d\bar{z} + (c\bar{k}e^{\bar{k}z}/\bar{\psi})dz = 0$$

on the contour γ . Here c is real and nonzero.

The zeroes of ψ' are singular points of this foliation. The manifold of zeroes $N_0 = \{\psi'(k, x, y) = 0\}$ is compact and two-dimensional $N_0 \subset \mathbb{C} \times \mathbb{R}^2 = \{k, x, y\}$. The intersection of N_0 with every plane $k = \text{const}$ consists of finite number of isolated points. The surface N_0 projects into the k -plane. The image is a compact set $N_0 \rightarrow N^* \subset \mathbb{C}$. We know that N^* is compact and does not cover 0. For all $k \in \mathbb{C} \setminus N^*$ our foliation on the 2-torus has no singular points. Our flow is analytic. Such flow is always homeomorphic

to the straight line flow with some "rotation number" $\rho(k)$ according to the famous classical results. So we have

$$\bar{\psi}'/\bar{k}e^{k\bar{z}}d\bar{z} + \psi'/ke^{kz}dz = 0.$$

Let us introduce a real function

$$F = \left(\frac{1}{|k|^2} + \sum_j^n \left(a_j \frac{e^{l_j\bar{z} - \bar{l}_j z}}{(k + l_j)(\bar{k} - \bar{l}_j)} + \bar{a}_j \frac{e^{-l_j\bar{z} + \bar{l}_j z}}{(k - l_j)(\bar{k} + \bar{l}_j z)} \right) \right) e^{k\bar{z} + \bar{k}z}.$$

Easy to check that $\partial F = 2\psi'/ke^{kz}$ and $\bar{\partial} F = 2\bar{\psi}'/\bar{k}e^{k\bar{z}}$. So for all $k \in \mathbb{C} \setminus N^*$ the contours are globally given by the equation $dF = 0$ or $F = \text{const}$ in \mathbb{R}^2 . For all values of $k \in \mathbb{C}$ the nonsingular levels $F = a$ define trajectories which do not approach singularity. All these trajectories are either compact and homotopic to zero or topologically equivalent in \mathbb{R}^2 to the straight line trajectory around the torus with some rotation number. All components of the level $F = \text{const}$ approaching singular points are either isolated singular points (the centers of foliation) or separatrices of the saddles (including degenerate saddles). So we proved the next statement:

Lemma 6 *Our foliation is equivalent to Hamiltonian system (foliation) with hamiltonian F in every compact domain E in the universal covering $E \subset \mathbb{R}^2$.*

As a corollary we conclude that

1. All singular points are either centers or saddles (maybe degenerate).
2. There are no limit cycles homotopic to zero in the torus, including limit cycles constructed from the pieces of separatrices from saddles to saddles.

Is our foliation globally topologically equivalent to the Hamiltonian foliation in the torus given by some multi-valued Hamiltonian (or closed one-form)?

Can limit cycles non-homotopic to zero exist for our system?

Let such cycle γ be given. It may be nonsingular or consists of several pieces of separatrices. Anyway, the value of function F along γ is constant, and the rotation number along γ is rational. According to our assumption, a family of non-closed open nonsingular trajectories γ_1 approaches γ asymptotically for $t \rightarrow \infty$. We can always choose γ_1 such that $F(\gamma) \neq F(\gamma_1)$.

There are two cases:

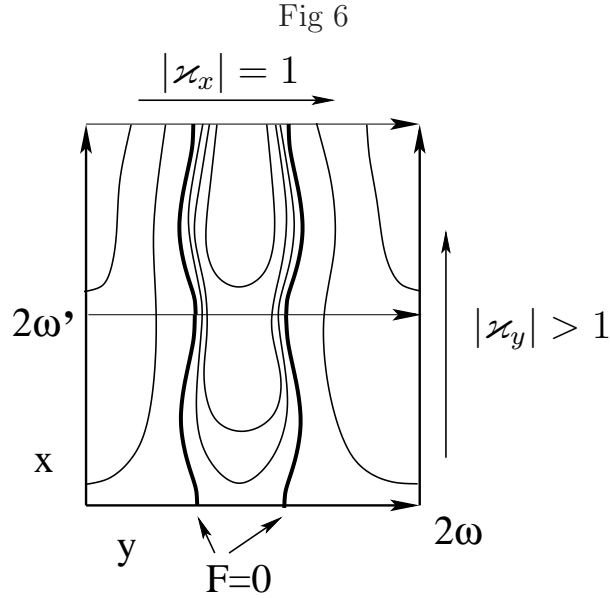
Case 1: The function ψ' is unbounded along the closed contour γ . It is possible only for such contours that $F(\gamma) = 0$ (see proof below). Any closed contour non-homotopic to zero in the torus with unbounded ψ' presents a limit cycle for our foliation—see Example to the Case 1 and Fig 6 below:

Example to the case 1: Let the functions c, F are the same as above. We take following parameters:

$$n = 2, l_1 = 1/2, l_2 = i/2, a_1 = a_2 = 0.2, k = 0.55i$$

$$c = 1 + 0.4 \cos(x) + 0.4 \cos(y)$$

Making numerical calculation, we are coming to the limit cycles γ where $b = [\gamma] \in H_1(T^2, \mathbb{Z})$ and b is a basic cycle along the y -axis (imaginary). We have here $F(\gamma) = 0$:



Here the contours $F = 0$ are closed and non-homotopic to 0. Bloch multiplier in the y -direction is different from 1. Therefore differential dF

exponentially increases in the y -direction, and all contours $F = c$ tend to the contour $F = 0$ as $y \rightarrow +\infty$

Case 2: The function ψ' is bounded along the open contours γ_1 .

According to the Lemma below it is always true if all connectivity components of the level $F = 0$ are compact in R^2 . The rotation number is rational for the closed contour γ non-homotopic to zero. So the trajectory γ corresponds to the shift $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by the lattice vector with unitary multiplier. After the number of iterations M we get rotation number for g^M equal to 1. But the form dF transforms as $g^*(dF) = \varkappa dF$ for all lattice shifts g with real Bloch multipliers $g^*\psi' = \varkappa\psi'$, $\varkappa \in \mathbb{R}$. We know that $F = \text{const}$ along γ_1 and that the form dF is invariant under the shift g^M with unit multiplier $\varkappa = 1$. So we see finally that $F(\gamma) = F(\gamma_1)$ for all γ_1 approaching γ if γ is nonsingular. This is the contradiction for the nonsingular γ because F is a smooth (even analytic) function. It is nonconstant along the transversal section to γ . It is true also for the singular γ constructed from separatrices. To prove this statement we should use specific Bloch properties of function F written by the explicit formula, instead of dF as above. In particular, we have to use its behavior under the lattice translations instead of the form dF .

So we conclude that our foliation cannot have nonsingular (and singular) limit cycles nonhomotopic to zero and such that $F(\gamma) \neq 0$.

Lemma 7 *Let $\hat{\gamma}$ be any open nonsingular trajectory such that topological closure of its projection γ in the torus T^2 does not contain the levels $F = 0$. Then ψ' is bounded along the contours γ . For every nonsingular periodic field $c \neq 0$ and $|k|$ big enough this condition is satisfied.*

Proof follows immediately from the formulas for the functions F and ψ' for $|k|$ big enough. The function ψ' looks like $e^{k\bar{z}}$ multiplied by the bounded periodic function. So $|\psi'| \sim e^{\text{Re}(k\bar{z})}$. The function F looks like a bounded periodic function multiplied by the function $e^{2\text{Re}(k\bar{z})}$. So the function $e^{\text{Re}(k\bar{z})}$ is bounded along the levels $F = \text{const} \neq 0$. Therefore ψ' is also bounded along this contour. For $|k| \rightarrow \infty$ big enough the asymptotic of function F looks like $(c/|k|^2)e^{k\bar{z}+\bar{k}z}$, so this function is separated from zero. Lemma is proved.

We are coming to the following

Conclusion. There are only three possibilities for the leaves of this foliation for any $k \in \mathbb{C}$:

1. There exists an open nonsingular trajectory γ for which $\psi'(k, x, y)$ is a solution to the self-adjoint d-bar boundary problem in the domain $D \subset T^2$ like cylinder, \hat{D} like strip and D_+ or D_- like warped half-planes. One should choose only one domain D_+ or D_- where $\psi'(k, x, y)/\sqrt{c}$ has decay at infinity transversal to the boundary (see Fig 4)
2. There exists a "maximal" cycle γ homotopic to zero in the torus T^2 consisting of separatrices of the saddles such that γ bounds the whole area of the torus (see Fig 5). ψ' gives solution to the self-adjoint boundary problem
3. There exists a limit cycle γ nonhomotopic to zero such that $F(\gamma) = 0$ (see Fig 6). For such boundary contour ψ' does not generate solution to the self-adjoint problem because it has exponential growth along the boundary.

This picture is a natural 2D generalization of one-dimensional case where functions $f(x^*) = \psi(\gamma_j(x), x^*)$ sitting in the gaps described an additional Dirichlet spectrum at the interval $x^* \in [x, x + T]$ and in the half-line $R^+ = [-\infty, x]$ or $R^- = [x, +\infty]$ for the same operator $L = -\partial_{x^*}^2 + u(x^*)$.

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